Algebraic geometry 2

Exercise Sheet 9

PD Dr. Maksim Zhykhovich Summer Semester 2026, 11.06.2026

Exercise 1. Let A be a ring and let $I \subset A$ be an ideal. Show that the morphism of affine schemes $\operatorname{Spec} A/I \to \operatorname{Spec} A$, induced by the canonical ring homomorphism $A \to A/I$, is a closed immersion.

Exercise 2. Let X be a scheme and $f \in \mathcal{O}_X(X)$. Recall that

$$X_f := \{x \in X \mid f_x \notin m_x \subset \mathcal{O}_{X,x}\}$$

is an open subset in X.
(1) Show that the image of f in $\mathcal{O}_X(X_f)$ by the restriction map is invertible.
(2) Deduce from (1) that the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X_f)$ can be extended to the ring homomorphism $\psi : \mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$.
(3) Assume that X is quasi-compact. Let $a \in \mathcal{O}_X(X)$ be an element, whose restriction to X_f is 0. Show that for some $n \geq 0$, $f^n a = 0$ in $\mathcal{O}_X(X)$. Deduce that the ring homomorphism ψ from (2) is injective.
Hint: Consider an open affine cover of X and use Remark 4.11(1) = Lemma 3.7 from the lecture.
(4) Assume that there is a finite open affine cover $X = \cup_{i=1}^r U_i$, such that every $U_i \cap U_j$ also admits a finite open affine cover. Let $b \in \mathcal{O}_X(X_f)$. Show that for some $n \geq 0$, $f^n_{|X_f} b$ lies in the image of the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X_f)$. Deduce that ψ from (2) is surjective and, thus, bijective.
Hint: Consider an open covering of X_f by $V_i := X_f \cap U_i$ and consider the family $\{b_{|V_i} \in \mathcal{O}_X(V_i)\}_{i=1,\dots,r}$ of restrictions of b. Lift these family (multiplying b by some f^n) to a family $\{a_i \in \mathcal{O}_X(U_i)\}_{i=1,\dots,r}$. Using that $\mathcal{O}_X$ is a sheaf, lift the obtained family (up to multiplication by some power of f) to an element $a \in \mathcal{O}_X(X)$.

Exercise 3. Let k be a field. Denote by $\mathbb{P}(k^n) = (k^{n+1} \backslash \{0\})/k^*$ the set-theoretical projective space in the classical sense and let
$X = \mathbb{P}^n_k := \operatorname{Proj}(k[X_0, \dots, X_n])$. Show that the following map is well-defined and bijective

$$\mathbb{P}(k^n) \to X(k),\ [\alpha_0 : \dots : \alpha_n] \mapsto \text{ homogeneous ideal } \langle (\alpha_i X_j - \alpha_j X_i)_{0 \leq i,j \leq n} \rangle,$$

where $X(k)$ denote the set of k-rational points of X.

Exercise 4. Let A be a ring and consider $X = \mathbb{P}^n_A$. Show that $\mathcal{O}_X(X) = A$.
Hint: Compare with Exercise 4, Sheet 6, from the last semester.